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Singularity-Free Breit Equation from Constraint Two-Body Dirac Equations

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Abstract

We examine the relation between two approaches to the quantum relativistic two-body problem: (1) the Breit equation, and (2) the two-body Dirac equations derived from constraint dynamics. In applications to quantum electrodynamics, the former equation becomes pathological if certain interaction terms are not treated as perturbations. The difficulty comes from singularities which appear at finite separations r in the reduced set of coupled equations for attractive potentials even when the potentials themselves are not singular there. They are known to give rise to unphysical bound states and resonances. In contrast, the two-body Dirac equations of constraint dynamics do not have these pathologies in many nonperturbative treatments. To understand these

marked differences we first express these contraint equations, which have an "external potential" form similar to coupled one-body Dirac equations, in a hyperbolic form. These coupled equations are then re-cast into two equivalent equations: (1) a covariant Breit-like equation with potentials that are exponential functions of certain "generator" functions, and (2) a covariant orthogonality constraint on the relative momentum. This reduction enables us to show in a transparent way that finite-r singularities do not appear as long as the the exponential structure is not tampered with and the exponential generators of the interaction are themselves nonsingular for finite r. These Dirac or Breit equations, free of the structural singularities which plague the usual Breit equation, can then be used safely under all circumstances, encompassing numerous applications in the fields of particle, nuclear, and atomic physics which involve highly relativistic and strong binding configurations.

I. INTRODUCTION

In contrast to the accepted description

$$[\gamma \cdot p + m + V]\psi = 0 \tag{1.1}$$

of the relativistic quantum mechanics of a single spin-one-half particle moving in an external potential V given by Dirac, a number of different approaches have been used for two interacting spin-one-half particles. A traditional one based on the Breit equation [1], also known as the Kemmer-Fermi-Yang equation [2],

$$[\boldsymbol{\alpha}_1 \cdot \boldsymbol{p}_1 + \beta_1 m_1 + \boldsymbol{\alpha}_2 \cdot \boldsymbol{p}_2 + \beta_2 m_2 + V(\boldsymbol{r}_{12})]\psi = E\psi, \tag{1.2}$$

contains a sum of single-particle Hamiltonians and an interaction term between them. (E is the total energy in an arbitrary frame.) Although the Breit equation is not manifestly covariant, it has provided good perturbative descriptions of the positronium and muonium energy levels. However, it is well known that those parts of the Breit interaction

$$V(\boldsymbol{r}) = \frac{-\alpha}{r}(1 - \frac{1}{2}\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 - \frac{1}{2}\boldsymbol{\alpha}_1 \cdot \hat{\boldsymbol{r}}\boldsymbol{\alpha}_2 \cdot \hat{\boldsymbol{r}}), \tag{1.3}$$

in the Breit equation beyond the Coulomb term should not be treated nonperturbatively, but must be handled only perturbatively. In other words, a consistent treatment of the Breit equation in powers of α will generate unwanted terms not present in quantum electrodynamics [Refs. 3,4].

A more serious difficulty of the Breit equation is that when the interactions are treated nonperturbatively, structural pole singularities could appear at finite r even when the interactions themselves are singularity-free there [5,8]. Two of us have found recently that these pole singularities occur under certain conditions depending on well-defined algebraic relations among the different potentials that could appear [6,7]. They lead to unphysical states or unphysical resonances and therefore must be strictly avoided [5,8].

The primary purpose of this paper is to show how these pole singularities can be avoided from the beginning so that the Breit equation can be used without difficulty in diverse

applications in particle, nuclear, and atomic physics involving highly relativistic motions and strong binding potentials. This is accomplished by relating this older approach to one that has been developed much more recently.

Dirac's constraint Hamiltonian dynamics [9] provides a framework for an approach proposed by Crater and Van Alstine [10,11] that differs notably from that of the Breit equation. It gives two-body coupled Dirac equations, each of which has an "external potential" form similar to the one-body Dirac equation in that for four-vector and scalar interactions one has

$$\mathcal{S}_1\psi \equiv \gamma_{51}[\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1]\psi = 0, \tag{1.4a}$$

$$\mathcal{S}_2\psi \equiv \gamma_{52}[\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2]\psi = 0, \tag{1.4b}$$

($\gamma_{5i},\ i = 1, 2$ are the γ_5 matrices for the constituent particles). Unlike the Breit approach, these equations are manifestly covariant and have interactions introduced by minimal substitution.

They have common solutions if the operators $\mathcal{S}_1, \mathcal{S}_2$ commute. This situation, called strong compatibility [9], can be achieved by the proper choice of the operators $\tilde{A}_i$ and $\tilde{S}_i$ [10,11]. The commutator cannot, however, be made to vanish for more general types of interactions such as pseudoscalar or pseudovector. Nevertheless, under certain circumstances the commutator can be reduced to a combination of $\mathcal{S}_1$ and $\mathcal{S}_2$ themselves. The equations are then said to be weakly compatible [9], because this will also ensure that solutions of $\mathcal{S}_1\psi = 0$ in the more general cases could be solutions of $\mathcal{S}_2\psi = 0$ as well. These compatibility properties are important because they guarantee the existence of common solutions to (i.e. the consistency of) the constraint equations before they are actually solved.

Although the constraint two-body (CTB) Dirac equations have been used far less frequently than the Breit equation, they have important advantages over the latter. In describing electromagnetic bound states [11,12] they yield nonperturbative and perturbative results in agreement with each other. That is, the exact (or numerical) solution produces

a spectrum that agrees through order α^4 with that given by perturbative treatment of the Darwin, spin-orbit, spin-spin, and tensor terms obtained from the Pauli reduction. In particular, total c.m. energies w for the e^+e^- system in the ${}^1J_J$ states is found to satisfy a Sommerfeld formula [11,12]

$$w = m\sqrt{2 + 2\Big/\left(1 + \frac{\alpha^2}{n + \sqrt{(J+\frac{1}{2})^2 - \alpha^2} - J - \frac{1}{2})^2}\right)}$$
$$= 2m - \frac{m\alpha^2}{4n^2} - \frac{m\alpha^4}{2n^3(2J+1)} + \frac{11}{64}\frac{m\alpha^4}{n^4} + O(\alpha^6). \tag{1.5}$$

They agree through order α^4 with those of the perturbative solution of the same equation, and also with those of standard approaches to QED. A recent paper has numerically extended this agreement at least to the $n = 1, 2, 3$ levels for all allowable J and unequal masses [12].

In this paper, we are concerned with another advantage of the CTB Dirac equations, namely that no unphysical states and resonances of the type discussed in [6,7] have ever appeared in past applications. We are able to show here that this is in fact true for a general interaction and that this is a consequence of the exponential structure of the interactions appearing in them. This result is obtained by first reducing the CTB Dirac equations to a Breit equation and an equation describing an orthogonality constraint. The equivalent Breit equation is then shown to be singularity-free, provided that the exponential interaction structure is not destroyed by inadvertent approximations and that the operators appearing in the exponent are themselves free of finite r singularities.

The exponential structure that tames the unphysical singularities turns out to be a consequence of a relativistic "third law" describing the full recoil effects between the two interacting particles. We carefully trace, in the formulation of the CTB Dirac equations, how this structure arises from the need to make these equations at least weakly compatible. Compared to the *laissez-aller* approach of the Breit equation for which any interaction seems possible, the restriction of the interaction structure needed in the constraint approach represents a conceptual advance in the problem. For this reason, we take pain to elucidate its conceptual foundation as we present elements of the CTB Dirac equations needed for our

demonstration that they are singularity-free.

This paper is organized as follows. We start in Sec. II with a brief overview of the derivation of the constraint two-body Dirac equations for scalar interactions both to define the notation used and to describe the main concepts involved. One of the most important properties is the compatibility of the two constraints. We remind the reader in Sec. III how to introduce general interactions into them that preserve this property. In Sec.IV we derive a covariant version of the Breit equation from the constraint equations, introducing the concept of exponential generators for a wide range of covariant interactions. In Sec. V we clarify the structure of this covariant Breit equation by decomposing it into a matrix form involving singlet and triplet components of the matrix wave function followed by reduction to radial form by the use of a vector spherical harmonic decomposition. This reveals clearly how the constraint approach avoids the structural pole singularities that have plagued the original Breit equation since its introduction over 65 years ago. In Sec. VI we show by contrast, how the pole singularities arises for each of the nonzero generators if one uses the original Breit interaction. Section VII contains brief concluding remarks.

II. TWO-BODY DIRAC EQUATIONS OF CONSTRAINT DYNAMICS

Following Todorov [13], we shall use the following dynamical and kinematical variables for the constraint description of the relativistic two-body problem:

i.) relative position, $x_1 - x_2$

ii.) relative momentum, $p = (\epsilon_2 p_1 - \epsilon_1 p_2)/w$,

iii.) total c.m. energy, $w = \sqrt{-P^2}$,

iv.) total momentum, $P = p_1 + p_2$,

and v.) constituent on-shell c.m. energies,

$$\epsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w}, \quad \epsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w}. \tag{2.1}$$

In terms of these variables, we have

$$p_1 = \epsilon_1 \hat{P} + p, \qquad p_2 = \epsilon_2 \hat{P} - p, \tag{2.2}$$

where $\hat{P} = P/w$.

We start from the (compatible) Dirac equations for two free particles

$$\mathcal{S}_{10}\psi = (\theta_1 \cdot p_1 + m_1\theta_{51})\psi = 0\,, \tag{2.3a}$$

$$S_{20}\psi = (\theta_2 \cdot p_2 + m_2\theta_{52})\psi = 0\,, \tag{2.3b}$$

where ψ is just the product of the two single-particle Dirac wave functions. These equations can be written as

$$\mathcal{S}_{10}\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51})\psi = 0\,, \tag{2.4a}$$

$$S_{20}\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52})\psi = 0, \tag{2.4b}$$

when expressed in terms of the Todorov variables. The "theta" matrices

$$\theta_i^\mu \equiv i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^\mu,\ \ \mu = 0, 1, 2, 3\ , i = 1, 2 \tag{2.5}$$

$$\theta_{5i} \equiv i\sqrt{\frac{1}{2}}\gamma_{5i} \tag{2.6}$$

satisfy the fundamental anticommutation relations

$$[\theta_i^\mu, \theta_i^\nu]_+ = -g^{\mu\nu}, \tag{2.7}$$

$$[\theta_{5i}, \theta_i^\mu]_+ = 0, \tag{2.8}$$

$$[\theta_{5i}, \theta_{5i}]_+ = -1. \tag{2.9}$$

[Projected "theta" matrices then satisfy

$$[\theta_i \cdot \hat{P}, \theta_i \cdot \hat{P}]_+ = 1, \tag{2.10}$$

$$[\theta_i \cdot \hat{P}, \theta_{i\perp}^{\mu}]_+ = 0, \tag{2.11}$$

where $\theta_{i\perp}^{\mu} = \theta_{i\nu}(\eta^{\mu\nu} + \hat{P}^{\mu}\hat{P}^{\nu})]$. They are modified Dirac matrices [14] which ensure that the Dirac operators $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$ are the square root operators of the corresponding mass-shell operators $-\frac{1}{2}(p_1^2 + m_1^2)$ and $-\frac{1}{2}(p_2^2 + m_2^2)$.

Using the Todorov variables and the above brackets, the difference

$$(\mathcal{S}_{10}^2 - \mathcal{S}_{20}^2)\psi = 0 = \frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)\psi \tag{2.12}$$

leads to an equation

$$P \cdot p\psi = \frac{1}{2}[w(\epsilon_1 - \epsilon_2) - (m_1^2 - m_2^2)]\psi = 0. \tag{2.13}$$

The physical significance of the orthogonality of these two momenta is to put a constraint on the relative momentum (eliminating the relative energy in the c.m. frame).

We will also use covariant (c.m. projected) versions of the Dirac α and β matrices here defined by

$$\beta_i = -\gamma_i \cdot \hat{P} = 2\theta_{5i}\theta_i \cdot \hat{P}, \tag{2.14}$$

$$\alpha_i^{\mu} = 2\theta_{i\perp}^{\mu}\theta_i \cdot \hat{P}, \tag{2.15}$$

and

$$\sigma_i^{\mu} = \gamma_{5i}\alpha_i^{\mu} = 2\sqrt{2}i\theta_{5i}\,\theta_i \cdot \hat{P}\,\theta_{\perp i}, \quad i = 1, 2. \tag{2.16}$$

These covariant Dirac matrices take on the simple form $\alpha_i^{\mu} = (0, \boldsymbol{\alpha}_i)$ and $\sigma_i^{\mu} = (0, \boldsymbol{\sigma}_i)$ in the center-of-mass system for which $\hat{P} = (1, \mathbf{0})$.

If we now introduce scalar interactions between these particles by naively making the minimal substitutions

$$m_i \to M_i = m_i + S_i,\ i = 1, 2 \tag{2.17}$$

(as done in the one-body equation), the resulting Dirac equations

$$\mathcal{S}_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51})\psi = 0, \tag{2.18a}$$

$$\mathcal{S}_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52})\psi = 0 \tag{2.18b}$$

will not be compatible because

$$[\mathcal{S}_1, \mathcal{S}_2]_-\psi = [\theta_1 \cdot p, M_2\theta_{52}] + [M_1\theta_{51}, -\theta_2 \cdot p]- = -i(\partial M_1 \cdot \theta_1\theta_{52} + \partial M_2 \cdot \theta_2\theta_{51})\psi \neq 0, \tag{2.19}$$

where ∂ is the four-gradient.

In the earlier work [10,15], compatibility is reinstated by generalizing the naive $\mathcal{S}_1$ and $\mathcal{S}_2$ operators with the help of supersymmetry arguments. The procedure contains four major steps:

a) Find the supersymmetries of the pseudoclassical limit of an ordinary free one-body Dirac equation.

b) Introduce interactions of a single Dirac particle with external potentials that preserve these supersymmetries. For scalar interactions, this requires the coordinate replacement

$$x^\mu \to \tilde{x}^\mu \equiv x^\mu + i\frac{\theta^\mu\theta_5}{m + S(\tilde{x})}. \tag{2.20}$$

(Note that the Grassmann variables satisfy $\theta^2 = 0$. As a result this self referent or recursive relation has a terminating Taylor expansion).

c) Maintain the one-body supersymmetries for each spinning particle through the replacement

$$(x_1 - x_2) \to (\tilde{x}_1 - \tilde{x}_2) \tag{2.21}$$

in the relativistic potentials S_i.

These steps lead to the pseudoclassical constraints (the weak equality sign $\approx$ means these equations are constraints imposed on the dynamical variables)

$$\mathcal{S}_1 = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51} - i\partial M_2/M_1 \cdot \theta_2\theta_{52}\theta_{51}) \approx 0, \tag{2.22a}$$

$$\mathcal{S}_2 = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52} + i\partial M_1/M_2 \cdot \theta_1\theta_{52}\theta_{51}) \approx 0. \tag{2.22b}$$

They are strongly compatible under the following two conditions:

i.) the mass potentials are related through a relativistic "third law"

$$\partial(M_1^2 - M_2^2) = 0, \tag{2.23}$$

and ii.) they depend on the separation variable only through the space-like projection perpendicular to the total momentum

$$M_i = M_i(x_\perp), \tag{2.24}$$

where

$$x_\perp^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu)(x_1 - x_2)_\nu. \tag{2.25}$$

Integration of the "third law" condition yields

$$M_1^2 - M_2^2 = m_1^2 - m_2^2 \tag{2.26}$$

with the hyperbolic solution

$$M_1 = m_1 \cosh L \, + m_2 \sinh L, \;\; M_2 = m_2 \cosh L \, + m_1 \sinh L, \tag{2.27}$$

given in terms of a single invariant function $L = L(x_\perp)$.

The $x_\perp$ dependence of the potential and the relativistic "third law" lie at the heart of two-body constraint dynamics. Without these conditions the constraints would not be compatible. While the physical importance of the $x_\perp$ dependence lies in its exclusion of the relative time in the c.m. frame, the "third law" condition relates the mutual interactions between the particles to the effective potentials each particle feels in the presence of the other in a consistent way. It is useful to show its implications in the simpler case of spinless particles. The two generalized mass shell constraints that are the counterparts of Eq. (2.22) for scalar interactions are

$$\mathcal{H}_i = p_i^2 + M_i^2 \approx 0, \qquad i = 1, 2. \tag{2.28}$$

The compatibility condition for these two constraints involves the classical Poisson bracket

$$[\mathcal{H}_1, \mathcal{H}_2] = [p_1^2, M_2^2] + [M_1^2, p_2^2] + [M_1^2, M_2^2] \approx 0. \tag{2.29}$$

One can see that this is satisfied provided that the "third law" condition Eq. (2.23) and condition (2.24) are satisfied. (Although the "third law" solution Eq. (2.26) combined with Eq. (2.24) is the simplest solution, it is not unique [15].)

For scalar interactions parametrized by

$$M_i = m_i + S_i, \quad i = 1, 2, \tag{2.30}$$

the "third law" condition becomes

$$m_1 S_1 = m_2 S_2, \tag{2.31}$$

in the nonrelativistic limit ($|S_i| << m_i$). This result can also be obtained from Eq. (2.27) by keeping only terms linear in L. The two constraints (2.28) can now be written as

$$p_i^2 + M_i^2 \approx p^2 + 2m_i S_i + S_i^2 + m_i^2 - \epsilon_i^2 = 0, \tag{2.32}$$

where we have used the fact that $\mathcal{H}_1 - \mathcal{H}_2 = P \cdot p \approx 0$ remains unchanged upon the introduction of scalar interaction in Eq. (2.24). Hence the total c.m. energy $w = \epsilon_1 + \epsilon_2$ takes on a familiar form in the nonrelativistic limit

$$w = m_1 + m_2 + \frac{p^2}{2\mu} + S + O(S^2), \tag{2.33}$$

where

$$S = (m_1 + m_2) S_1 / m_2 = (m_1 + m_2) S_2 / m_1. \tag{2.34}$$

d) The final step is to canonically quantize the classical dynamical system defined by $\mathcal{S}_1$ and $\mathcal{S}_2$ by replacing the Grassmann variables $\theta_{\mu i}, \theta_{5i},\ i = 1, 2$ with two mutually commuting sets of theta matrices, and the position and coordinate variables by operators satisfying the fundamental commutation relation

$$\{x^\mu, p^\nu\} \rightarrow [x^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{2.35}$$

The compatible pseudoclassical spin constraints $\mathcal{S}_1$ and $\mathcal{S}_2$ then become commuting quantum operators

$$[\mathcal{S}_1, \mathcal{S}_2] = 0. \tag{2.36}$$

The resulting CTB Dirac equations for scalar interactions

$$\mathcal{S}_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51} - i\partial L \cdot \theta_2\theta_{52}\theta_{51})\psi = 0, \tag{2.37a}$$

$$\mathcal{S}_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52} + i\partial L \cdot \theta_1\theta_{52}\theta_{51})\psi = 0, \tag{2.37b}$$

where

$$\partial L = \frac{\partial M_1}{M_2} = \frac{\partial M_2}{M_1}, \tag{2.38}$$

are then said to be strongly compatible. This strong compatibility has been achieved by a supersymmetry which produces the extra spin-dependent recoil terms involving ∂L. These extra terms vanish, however, when one of the particles becomes infinitely massive (as seen by the parametrization $M_i = m_i + S_i$ of the scalar potential) so that we recover the expected one-body Dirac equation in an external scalar potential.

Note that the Dirac constraint operators satisfy [10]

$$\mathcal{S}_1^2 - \mathcal{S}_2^2 = -\frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2) = -P \cdot p \approx 0. \tag{2.39}$$

Thus the relative momentum remains orthogonal to the total momentum after the introduction of the interaction. This also implies that the constituent on-shell c.m. energies ϵ_i are weakly equal to their off mass shell values ($-\hat{P} \cdot p_i \approx \epsilon_i$). Notice further that since $[P \cdot p, M(x_\perp)] \sim P \cdot x_\perp \equiv 0$, this constraint does not violate the requirement of compatibility given in Eqs. (2.24-25).

III. A GENERAL INTERACTION FOR TWO-BODY DIRAC EQUATIONS

The previous work [16] has shown how the compatibility problem can be solved without having to invent new supersymmetries if the scalar potential is replaced by vector, pseudoscalar, pseudovector, or tensor potentials. That work also relates the supersymmetric or

"external potential" approach to the alternative treatment of the two body Dirac equations of constraint dynamics presented by H. Sazdjian [17].

The "external potential" form Eqs. (2.37) of the CTB Dirac equations for scalar interaction can be rewritten in the hyperbolic form [16]

$$\mathcal{S}_1\psi = (\cosh\Delta\ \mathbf{S}_1 + \sinh\Delta\ \mathbf{S}_2)\psi = 0, \tag{3.1a}$$

$$\mathcal{S}_2\psi = (\cosh\Delta\ \mathbf{S}_2 + \sinh\Delta\ \mathbf{S}_1)\psi = 0, \tag{3.1b}$$

where Δ generates the scalar potential terms in (2.37) provided that

$$\Delta = -\theta_{51}\theta_{52}L(x_\perp). \tag{3.2}$$

The operators $\mathbf{S}_1$ and $\mathbf{S}_2$ are auxiliary constraints of the form

$$\mathbf{S}_1\psi \equiv (\mathcal{S}_{10}\cosh\Delta\ \ + \mathcal{S}_{20}\sinh\Delta\ )\psi = 0, \tag{3.3a}$$

$$\mathbf{S}_2\psi \equiv (\mathcal{S}_{20}\cosh\Delta\ \ + \mathcal{S}_{10}\sinh\Delta\ )\psi = 0. \tag{3.3b}$$

To verify that the "external potential" forms Eq. (2.37) result from using Eqs. (3.3) in Eqs. (3.1), one simply commutes the free Dirac operator $\mathcal{S}_{i0}$ to the right onto the wave function using Eqs. (2.7-2.11), (2.38) and hyperbolic identities [16]. With this construction, the interaction enters only through an invariant matrix function Δ with all other spin-dependence a consequence of the factors contained in the kinetic free Dirac operators $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$.

Even though the form of the contraints Eqs. (3.1) and (3.3) were motivated by examining world scalar interactions, let us propose them for arbitrary Δ and determine their compatibility requirements. We do this for arbitrary interactions by generalizing arguments given in Refs.[16-17]. First consider the conditions for the compatibility of Eqs. (3.3a-b). Multiplying Eq. (3.3a) by $\mathcal{S}_{10}$ and Eq. (3.3b) by $\mathcal{S}_{20}$ and subtracting we obtain

$$P\cdot p\ (\cosh\Delta)\ \psi = 0. \tag{3.4a}$$

Multiplying Eq. (3.3b) by $\mathcal{S}_{10}$ and Eq. (3.3a) by $\mathcal{S}_{20}$ and subtracting we obtain

$$P \cdot p \; (\sinh\, \Delta) \; \psi = 0. \tag{3.4b}$$

We have used Eq. (2.2) and $\epsilon_1 - \epsilon_2 = (m_1^2 - m_2^2)/w$ to simplify these equations. Multiplying Eq. (3.4a) by sinhΔ, Eq. (3.4b) by coshΔ, bringing the operator $P \cdot p$ to the right and subtracting we find the condition

$$[P \cdot p, \Delta]\psi = 0. \tag{3.5}$$

Multiplying Eq. (3.4a) by coshΔ, Eq. (3.4b) by sinhΔ, bringing the operator $P \cdot p$ to the right and subtracting we find the further condition

$$P \cdot p \; \psi = 0. \tag{3.6}$$

Notice that this latter condition was previously associated with the "third law" condition when derived from the "external potential" forms of the constraints (see Eq. (2.39)). Here the "third law" condition is built into the constraint by having the same generator Δ for Eqs. (3.3a) and (3.3b). Thus the two tentative constraints Eqs. (3.3a) and (3.3b) taken together imply that for arbitrary Δ the orthogonality condition $P \cdot p \approx 0$ has to be satisfied when acting on ψ. However, in order to verify that there are no additional conditions beyond Eqs. (3.5) and (3.6) we must check for mathematical consistency by examining the compatibility condition. We compute the commutator $[\mathbf{S}_1, \mathbf{S}_2]$ by rearranging its eight terms and find that

$$\begin{aligned}[\mathbf{S}_1, \mathbf{S}_2] = [\mathcal{S}_{10}, \mathrm{cosh}\Delta]\mathbf{S}_2 - [\mathcal{S}_{20}, \mathrm{cosh}\Delta]\mathbf{S}_1 + [\mathcal{S}_{20}, \mathrm{sinh}\Delta]\mathbf{S}_2 - [\mathcal{S}_{10}, \mathrm{sinh}\Delta]\mathbf{S}_1 \\ + \mathrm{cosh}\Delta(\mathcal{S}_{10}^2 - \mathcal{S}_{20}^2)\mathrm{sinh}\Delta - \mathrm{sinh}\Delta(\mathcal{S}_{10}^2 - \mathcal{S}_{20}^2)\mathrm{cosh}\Delta\end{aligned} \tag{3.7}$$

does not in general vanish, unlike Eqs. (2.36) and (2.37). By using Eqs. (2.12) and bringing the operator $P \cdot p$ to the right, and using the conditions given in Eqs. (3.5) and (3.6) we can reduce $[\mathbf{S}_1, \mathbf{S}_2]\psi$ to only terms involving $\mathbf{S}_1\psi$ and $\mathbf{S}_2\psi$. Since Eqs.(3.5) and (3.6) follow from combining the constraints $\mathbf{S}_i\psi = 0$, no further conditions for mathematical consistency need be imposed on the constraints or the wave function. Eq. (3.6) is the quantum counterpart of

Eq. (2.39) but for arbitrary interactions. Eq. (3.5) is also satisfied for arbitrary Δ provided only that the generator Δ satisfies

$$\Delta = \Delta(x_\perp) \tag{3.8}$$

generalizing Eq. (2.24).

The weak compatibility of the "external potential" form of the constraints Eq. (3.1) for general Δ

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0 \tag{3.9}$$

can be seen by examining the four commutators in $[\mathcal{S}_1, \mathcal{S}_2]$. The commutator

$$[\cosh\Delta\ \mathbf{S}_1, \cosh\Delta\ \mathbf{S}_2] = \cosh\Delta(\cosh\Delta[\mathbf{S}_1, \mathbf{S}_2] + [\mathbf{S}_1, \cosh\Delta]\mathbf{S}_2 + [\cosh\Delta, \mathbf{S}_2]\mathbf{S}_1) \tag{3.10}$$

and is weakly zero since $\mathbf{S}_i\psi = 0$ and $[\mathbf{S}_1, \mathbf{S}_2]\psi = 0$. Likewise, we have

$$[\sinh\Delta\mathbf{S}_2, \sinh\Delta\mathbf{S}_1]\psi = 0. \tag{3.11}$$

The remaining two brackets are

$$\begin{aligned} &- [\cosh\Delta\ \mathbf{S}_1, \sinh\Delta\ \mathbf{S}_1] - [\sinh\Delta\ \mathbf{S}_2, \cosh\Delta\ \mathbf{S}_2] \\ &= [\mathbf{S}_1, \sinh(\Delta)]\mathbf{S}_1 - \sinh(\Delta)[\cosh(\Delta), \mathbf{S}_1]\mathbf{S}_1 + (1 \to 2). \end{aligned} \tag{3.12}$$

Since they contain the constraints on the right we obtain Eq. (3.9) after combining with Eqs. (3.10) and (3.11).

One final feature should be mentioned. Eqs. (3.1) and (3.3) are also applicable for a sum of the four "polar" interactions

$$\Delta_1 = \Delta_L + \Delta_J + \Delta_{\mathcal{G}} + \Delta_{\mathcal{F}}, \tag{3.13}$$

where

$$\Delta_L = -L\theta_{51}\theta_{52} \quad \text{scalar}, \tag{3.14}$$

$$\Delta_J = J\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \quad \text{time}-\text{like vector}, \tag{3.15}$$

$$\Delta_{\mathcal{G}} = \mathcal{G}\theta_{1\perp}\cdot\theta_{2\perp} \quad \text{space}-\text{like vector}, \tag{3.16}$$

$$\Delta_{\mathcal{F}} = 4\mathcal{F}\theta_{1\perp}\cdot\theta_{2\perp}\theta_{51}\theta_{52}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \quad \text{tensor (polar)}. \tag{3.17}$$

For the sum

$$\Delta_2 = \Delta_C + \Delta_H + \Delta_I + \Delta_Y \tag{3.18}$$

of their axial counterparts

$$\Delta_C = -C/2, \qquad \text{pseudoscalar}, \tag{3.19}$$

$$\Delta_H = -2H\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2\theta_{51}\theta_{52} \qquad \text{time}-\text{like pseudovector}, \tag{3.20}$$

$$\Delta_I = 2I\theta_{1\perp}\cdot\theta_{2\perp}\theta_{51}\theta_{52} \qquad \text{space}-\text{like pseudovector}, \tag{3.21}$$

$$\Delta_Y = -2Y\theta_{1\perp}\cdot\theta_{2\perp}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \qquad \text{tensor (axial)}, \tag{3.22}$$

the $\sinh\Delta$ terms in Eq. (3.1) should carry negative signs instead [16]. In contrast, Eq. (3.3) as written remains valid as is for Δ_2. For systems with both polar and axial interactions [16], one uses $\Delta_1 - \Delta_2$ in (3.1) and $\Delta_1 + \Delta_2$ in (3.3). The terms $L, J, \mathcal{G}, \mathcal{F}, C, H, I, Y$ are arbitrary invariant functions of $x_\perp$.

Eq. (3.3) is more convenient to use for the construction of Breit-like equations from the constraint formalism for general interactions. Eq. (3.1) is more convenient if one aims to obtain a set of Dirac equations in an "external potential" form similar to that exhibited in the one body Dirac equation with the transformation properties one would expect for such potentials. We have already seen this for the scalar case in which the scalar interaction "generator" L in (3.14) in the hyperbolic form leads to a modification of the mass term. In

the case of combined scalar, time- and space-like vector and pseudoscalar interactions, we use the hyperbolic parametrization

$$M_1 = m_1 \cosh L + m_2 \sinh L\,, \tag{3.23a}$$

$$M_2 = m_2 \cosh L + m_1 \sinh L\,, \tag{3.23b}$$

$$E_1 = \epsilon_1 \cosh J + \epsilon_2 \sinh J\,, \tag{3.24a}$$

$$E_2 = \epsilon_2 \cosh J + \epsilon_1 \sinh J\,, \tag{3.24b}$$

$$G = e^{\mathcal{G}}, \tag{3.25}$$

where L, J, and $\mathcal{G}$ generate scalar, time-like vector and space-like vector interactions respectively, while C generates pseudoscalar interactions. The resultant "external potential" form for

$$\Delta = \Delta_J + \Delta_L + \Delta_{\mathcal{G}} + \Delta_C \tag{3.26}$$

is

$$\begin{aligned}\mathcal{S}_1\psi = (&G\theta_1 \cdot p + E_1\theta_1 \cdot \hat{P} + M_1\theta_{51}\\ &+iG(\theta_2 \cdot \partial\mathcal{G}\theta_{1\perp} \cdot \theta_{2\perp} + \theta_2 \cdot \partial J\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} - \theta_2 \cdot \partial L\theta_{51}\theta_{52} + \theta_2 \cdot \partial C/2))\psi = 0\end{aligned} \tag{3.27a}$$

$$\begin{aligned}\mathcal{S}_2\psi = (&-G\theta_2 \cdot p + E_2\theta_2 \cdot \hat{P} + M_2\theta_{52}\\ &-iG(\theta_1 \cdot \partial\mathcal{G}\theta_{1\perp} \cdot \theta_{2\perp} + \theta_1 \cdot \partial J\theta_1 \cdot \hat{P}\theta_2\hat{P} - \theta_1 \cdot \partial L\theta_{51}\theta_{52} + \theta_1 \cdot \partial C/2))\psi = 0.\end{aligned} \tag{3.27b}$$

The scalar generator produces the mass or scalar potential M_i terms, the time-like vector generator produces the energy or time-like potential E_i terms, the space-like vector generator produces the transverse or space-like momentum G terms, while the pseudoscalar generator

produces only spin-dependent terms. Note that the vector and scalar interactions also have additional spin-dependent recoil terms essential for compatibility. The above features are just what one would expect for interactions transforming in this way and are direct consequences of the hyperbolic parametrization of the contraints. Other parameterizations may not have this important property.

IV. REDUCTION TO A BREIT EQUATION

We now derive a Breit equation from the CTB Dirac equations (3.3). Consider the combination $2(\theta_1 \cdot \hat{P}\mathbf{S}_1 + \theta_2 \cdot \hat{P}\mathbf{S}_2)$. The terms proportional to $w = \epsilon_1 + \epsilon_2$ are

$$w[\cosh\Delta + 2(\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P})\sinh\Delta] = w\ exp(\mathcal{D}) \tag{4.1}$$

where

$$\mathcal{D} = 2(\theta_1 \cdot \hat{P} \ \ \theta_2 \cdot \hat{P})\Delta. \tag{4.2}$$

The other terms can also be written in terms of $\mathcal{D}$, using the relations

$$\cosh\mathcal{D} = \cosh\Delta, \ \ \sinh\mathcal{D} = 2\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P}\sinh\Delta, \tag{4.3}$$

$$(2\theta_2 \cdot \hat{P}\theta_1 \cdot p - 2\theta_1 \cdot \hat{P}\theta_2 \cdot p)\sinh\Delta = -(2\theta_1 \cdot \hat{P}\theta_1 \cdot p - 2\theta_2 \cdot \hat{P}\theta_2 \cdot p)\sinh\mathcal{D}, \tag{4.4}$$

and

$$(2\theta_2 \cdot \hat{P}\theta_{51}m_1 + 2\theta_1 \cdot \hat{P}\theta_{52}m_2)\sinh\Delta = -(2\theta_1 \cdot \hat{P}\theta_{51}m_1 + 2\theta_2 \cdot \hat{P}\theta_{52}m_2)\sinh\mathcal{D}. \tag{4.5}$$

Then the combination $2(\theta_1 \cdot \hat{P}\mathbf{S}_1 + \theta_2 \cdot \hat{P}\mathbf{S}_2)$ of the two hyperbolic equations in Eq. (3.3) takes the simple form

$$w\ exp(\mathcal{D})\psi = (H_{10} + H_{20})exp(-\mathcal{D})\psi \tag{4.6}$$

where

$$H_{10} = -2\theta_1 \cdot \hat{P}\theta_1 \cdot p - 2\theta_1 \cdot \hat{P}\theta_{51}m_1 = \alpha_1 \cdot p_\perp + \beta_1 m_1 + \hat{P} \cdot p \tag{4.7}$$

$$H_{20} = 2\theta_2 \cdot \hat{P}\theta_2 \cdot p - 2\theta_2 \cdot \hat{P}\theta_{52}m_2 = -\alpha_2 \cdot p_\perp + \beta_2 m_2 - \hat{P} \cdot p \tag{4.8}$$

(with the definition $p_\perp \equiv p + \hat{P}p \cdot \hat{P}$) are covariant free Dirac Hamiltonians involving the covariant α and β matrices given previously in Eqs. (2.14) and (2.15). If we take

$$\Psi = exp(-\mathcal{D})\psi \tag{4.9}$$

we obtain finally the manifestly covariant Breit-like equation

$$w\ exp(2\mathcal{D})\Psi = (H_{10} + H_{20})\Psi. \tag{4.10}$$

The interactions appear in the Breit equation in the exponentiated form $exp(2\mathcal{D})$, where $2\mathcal{D}$ contains all eight interactions shown in Δ_1 and Δ_2. We can even add momentum-independent interactions proportional to $\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}$ to $2\mathcal{D}$ to give the more general interaction

$$\begin{aligned} 2\mathcal{D} = J - \beta_1\beta_2 L + \rho_1\rho_2 C + \gamma_{51}\gamma_{52}H + \sigma_1 \cdot \sigma_2(-I + \beta_1\beta_2 Y + \rho_1\rho_2\mathcal{F} + \gamma_{51}\gamma_{52}\mathcal{G}) \\ +\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}(N + \beta_1\beta_2 T + \rho_1\rho_2 S + \gamma_{51}\gamma_{52}R), \end{aligned} \tag{4.11}$$

where

$$\rho_i = \beta_i\gamma_{5i}, \tag{4.12}$$

the covariant σ is from Eq. (2.16), $\hat{r} = x_\perp/|x_\perp|$, and N, T, S, and R are arbitrary invariant functions of $x_\perp$. Note that each term in $2\mathcal{D}$ involves identical operators for particle 1 and particle 2. As a result, they all commute with each other. For example, we have

$$[\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}, \quad \sigma_1 \cdot \sigma_2] = 0\,. \tag{4.13}$$

Hence the a single exponential function $2\mathcal{D}$ can also be written as a product of separate exponentials.

Before continuing our discussion on the structure of the covariant Breit equation, we consider the remaining linear combination of the two constraint equations (3.3) involving

the difference $2(\theta_1 \cdot \hat{P}\mathbf{S}_1 - \theta_2 \cdot \hat{P}\mathbf{S}_2)$. Using identities in the beginning of this section we find that the difference equation becomes

$$(\epsilon_1 - \epsilon_2)exp(-\mathcal{D})\psi = (H_{10} - H_{20})exp(\mathcal{D})\psi\,. \tag{4.14}$$

Transforming to Ψ and using the Breit equation (4.10) gives

$$(\epsilon_1 - \epsilon_2)\Psi = \frac{(H_{10}^2 - H_{20}^2)}{w}\Psi = \frac{(m_1^2 - m_2^2)}{w}\Psi + \frac{2(H_{10} + H_{20})P \cdot p}{w}\Psi. \tag{4.15}$$

Combined with Eq. (2.13) this gives

$$P \cdot p\Psi = 0, \tag{4.16}$$

confirming the expectation that these momenta remain orthogonal after the interaction is introduced. This result has also been obtained recently by Mourad and Sazdjian [18] who emphasize that this would ensure the Poincaire' invariance of the theory. They further point out that the c.m. energy dependence of the potential in the "main equation" (our Eq. (4.10)) ensures the global charge conjugation symmetry [18] of the system, a feature that is missing from the original Breit equation.

In summary, the constraint equations imply covariant Breit-like equations of a certain form (4.10) whose wave function also satisfies the constraint equation (4.16). Alternatively, if we start off with a covariant Breit-like equation of the form (4.10) with $\mathcal{D} = \mathcal{D}(x_\perp)$ and require simultaneously that $P \cdot p\Psi = 0$, we can work backward to obtain the two compatible constraint equations (3.3).

We point out finally that the application of the constraint equations to electromagnetic interactions does not involve the term $\alpha_1 \cdot \hat{r}\alpha_2 \cdot \hat{r}$ appearing in the Breit interaction Eq. (1.2). Nevertheless, it does produce the correct spectrum, as shown in [11,12] using the "external potential" form in Eqs. (3.27) of the two-body Dirac equations with $L = C = 0$, $J = -\mathcal{G}$. As has been recently noted [19], in the context of the Breit-like form Eq. (4.10) of that equation, one obtains the reduction

$$(H_{10} + H_{20} + V_1 + V_2\alpha_1 \cdot \alpha_2 + V_3\gamma_{51}\gamma_{52} + V_4\sigma_1 \cdot \sigma_2)\Psi = w\Psi, \tag{4.17}$$

which contain pseudovector terms in place of the vector Breit term $(\alpha_1 \cdot \hat{r}\alpha_2 \cdot \hat{r})$.

V. STRUCTURE OF THE BREIT EQUATION

The Breit equation (4.10) can be written, as usual, as a set of coupled equations for different components of the wave function contained in Ψ. We work in the center-of-mass system for which $\hat{P} = (1, \mathbf{0})$, $\sigma = (0, \boldsymbol{\sigma})$, and $\hat{r} = (0, \hat{\boldsymbol{r}})$. We begin by simplifying the general interaction (4.11) to the more compact form

$$2\mathcal{D} = \sum_{\nu=0}^{3} (A_\nu + \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} B_\nu + \boldsymbol{\sigma_1} \cdot \hat{\boldsymbol{r}}\, \boldsymbol{\sigma_2} \cdot \hat{\boldsymbol{r}}\, C_\nu) q_\nu^{(1)} q_\nu^{(2)} \tag{5.1}$$

where the superscripts (1) and (2) label the interacting particles 1 and 2. The operators

$$(q_0, q_1, q_2, q_3) = (1, \gamma_5, -i\rho, \beta) \tag{5.2}$$

are defined so that q_1, q_2 and q_3 are analogous to the Pauli matrices σ_1, σ_2 and σ_3 satisfying

$$q_i q_j = \delta_{ij} + i\epsilon_{ijk} q_k \tag{5.3}$$

where i, j and $k = 1$, 2, and 3. Eq. (5.1) is in the form of four-scalar products

$$2\mathcal{D} = (A + \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} B + \boldsymbol{\sigma_1} \cdot \hat{\boldsymbol{r}}\, \boldsymbol{\sigma_2} \cdot \hat{\boldsymbol{r}}\, C) \cdot Q \tag{5.4}$$

involving the "four-vector" $Q_\nu = q_\nu^{(1)} q_\nu^{(2)}$, and

$$A = (J, H, -C, -L),\ \ B = (-I, \mathcal{G}, -\mathcal{F}, Y),\ \ C = (N, R, -S, T). \tag{5.5}$$

The wave function Ψ in Eq. (4.10) can be written as a spinor or column vector with two indices, one for each particle

$$\Psi = \Psi^{(1)} \Psi^{(2)}. \tag{5.6}$$

It is however more convenient to express the content of the wave function Ψ in terms of a new 4×4 matrix wave function Ψ'

$$\Psi' = \Psi^{(1)} \Psi^{(2)T} \alpha_y\,, \tag{5.7}$$

where $\Psi^{(2)}$ has been transposed and an operator α_y has been added on the right, as explained below. We can represent the operation of $\mathcal{A}^{(1)}$ for particle 1 and $\mathcal{B}^{(2)}$ for particle 2 acting

on the original spinor wave function Ψ in terms of operations $\mathcal{A}$ and $\mathcal{B}'$ on the new wave function Ψ',

$$\mathcal{A}^{(1)}\mathcal{B}^{(2)}\Psi \to \mathcal{A}\Psi^{(1)}[\mathcal{B}\Psi^{(2)}]^T\alpha_y = \mathcal{A}\Psi'\mathcal{B}', \tag{5.8}$$

where the matrix operator $\mathcal{B}'$ is

$$\mathcal{B}' = \alpha_y\mathcal{B}^T\alpha_y\,. \tag{5.9}$$

The arrow in Eq. (5.8) indicates the transformation of the operation of $\mathcal{A}^{(1)}\mathcal{B}^{(2)}$ on the wave function Ψ to the operation of $\mathcal{A}$ and $\mathcal{B}'$ with respect to the new wave function Ψ'. The operator α_y in Eq. (5.7) insures that operators such as $\boldsymbol{\alpha}^{(2)}$ acting on the second particle is represented by

$$\boldsymbol{\alpha}^{(2)}\Psi \to \Psi'\alpha_y\boldsymbol{\alpha}^T\alpha_y = -\Psi'\boldsymbol{\alpha}\,, \tag{5.10}$$

where the same negative sign appears for the different components of the operator $\boldsymbol{\alpha}$. By using the wave function Ψ' and Eq. (5.8), an operator acting on the first particle appears on the left side of the wave function Ψ', while an operator $\mathcal{O}$ acting on the second particle becomes $\alpha_y\mathcal{O}^T\alpha_y$ and appears on the right side of Ψ'.

In this matrix notation, the righthand side (RHS) of Eq. (4.10) becomes

$$(H_{10}+H_{20})\Psi \to \boldsymbol{p}\cdot\boldsymbol{\alpha}\,\Psi' + \boldsymbol{p}\cdot\Psi'\,\boldsymbol{\alpha} + m_1\beta\Psi' - m_2\Psi'\beta \equiv \mathrm{RHS}\,. \tag{5.11}$$

The reduction of Eq. (4.10) is facilitated by separating the matrix wave function into two parts:

$$\Psi = \Psi_0 + \Psi_1 \to \Psi' = {\Psi_0}' + {\Psi_1}' \equiv \sum_{\lambda,\kappa=0}^{3} q_\kappa\sigma_\lambda\psi_{\kappa\lambda}, \tag{5.12}$$

where $\sigma_0 = 1$, $\boldsymbol{\sigma} = \gamma_5\boldsymbol{\alpha}$ [20],

$$\Psi_0' = \sum_{\kappa=0}^{3} q_\kappa\psi_{\kappa 0}, \tag{5.13a}$$

and

$$\Psi_1' = \sum_{\kappa=0}^{3} q_\kappa \boldsymbol{\sigma} \cdot \boldsymbol{\psi}_\kappa = \sum_{\kappa=0}^{3} \sum_{\lambda=1}^{3} q_\kappa \sigma_\lambda \psi_{\kappa\lambda}. \tag{5.13b}$$

These parts give different results when operated on by $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$,

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \, \Psi \to -\boldsymbol{\sigma} \cdot \Psi' \boldsymbol{\sigma} = -3\Psi_0' + \Psi_1'. \tag{5.14}$$

This shows that $\Psi_0(\Psi_0')$ and $\Psi_1(\Psi_1')$ are respectively the spin-singlet and spin-triplet parts of the wave function. Furthermore, all the operators in Eq. (4.11) commute. With $(\boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}} \, \boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}})^2 = 1$, the tensor interaction can be written as

$$e^{\boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}} \, \boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}} \, C \cdot Q} = \cosh(C \cdot Q) + \boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}} \, \boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}} \, \sinh(C \cdot Q) \, . \tag{5.15}$$

This means that interaction (4.11) alone will not mix spin-singlet and spin-triplet states (as seen below in Eq.(5.34), the kinetic energy term does mix singlet and triplet states).

The reduction to the matrix form is easier for the spin-singlet wave function $\Psi_0 = \frac{1}{4}(1 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\Psi$ because of its simpler spin structure:

$$e^{(A + \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} B + \boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}} \, \boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}} \, C) \cdot Q} \Psi_0 = e^{T \cdot Q} \Psi_0, \tag{5.16}$$

where $T = A - 3B - C$. A Taylor expansion of the expontial operator on the above equation shows that it is necessary to evaluate the basic operation of the type

$$T \cdot Q \Psi_0 = \sum_{\nu=0}^{3} T_\nu q_\nu^{(1)} q_\nu^{(2)} \Psi_0. \tag{5.17}$$

In terms of the new wave function Ψ_0' of Eq. (5.13a), the above equation can be represented in the matrix form:

$$\sum_{\kappa=0}^{3} \sum_{\nu=0}^{3} T_\nu (q_\nu q_\kappa q_\nu') \psi_{\kappa 0} = \sum_{\kappa=0}^{3} T \cdot S_\kappa q_\kappa \psi_{\kappa 0} \, , \tag{5.18}$$

where $q_\nu' = \alpha_y q_\nu^T \alpha_y = \epsilon_\nu q_\nu$, with $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) = (1, 1, 1, -1)$ and we have introduce the quantity $(S_\kappa)_\nu$ defined by

$$q_\nu q_\kappa q_\nu' = (S_\kappa)_\nu q_\kappa, \tag{5.19a}$$

and

$$(S_\kappa)_\nu = \epsilon_\nu[1 + 2(1-\delta_\kappa 0)(1-\delta_{\nu 0})(\delta_{\kappa\nu}-1)]\,. \tag{5.19b}$$

The 4×4 matrix S is called a signature matrix because its matrix elements can only be $+1$ or -1. The row vectors of S are

$$\begin{aligned} S_0 &= (1,\ 1,\ 1, -1), \\ S_1 &= (1,\ 1, -1,\ 1), \\ S_2 &= (1, -1,\ 1,\ 1), \\ S_3 &= (1, -1, -1, -1)\,. \end{aligned} \tag{5.20}$$

Eq. (5.18) can be applied repeatedly to give the desired result

$$\begin{aligned} we^{T\cdot Q}\Psi_0 \;&=\; w\sum_{n=0}^{\infty}\frac{1}{n!}(T\cdot Q)^n\Psi_0 \\ &\to w\sum_{\kappa=0}^{3} e^{T\cdot S_\kappa} q_\kappa \psi_{\kappa 0} \equiv \mathrm{LHS}_0. \end{aligned} \tag{5.21}$$

It can also be used to prove the general result

$$f(T\cdot Q)\Psi_0 \to \sum_{\kappa=0}^{3} f(T\cdot S_\kappa) q_\kappa \psi_{\kappa 0}. \tag{5.22}$$

The treatment of the spin-triplet expression

$$e^{2\mathcal{D}}\Psi_1 = e^{(A+B)\cdot Q}[\cosh(C\cdot Q) + \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\,\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}\ \sinh(C\cdot Q)]\Psi_1 \tag{5.23}$$

is simplified by noting that the q_κ and $\boldsymbol{\sigma}$ matrices are independent of each other. Hence the Q dependences can be eliminated in favor of the signature vector S_κ with the help of Eq. (5.19a), which also applies to the q structure of Ψ_1. This leaves the spin-dependent part which has the form

$$\begin{aligned} \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\,\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}\Psi_1 &\to -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\Psi_1{}'\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} \\ &= \sum_{\kappa=0}^{3} q_\kappa(\boldsymbol{\sigma}\cdot\boldsymbol{\psi}_\kappa - 2\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\psi_\kappa})\,. \end{aligned} \tag{5.24}$$

Hence, using Eq. (5.15), the spin-triplet part of the left hand side of Eq. (4.10) is

$$\mathrm{LHS}_1 = w \sum_{\kappa=0}^{3} e^{(A+B)\cdot S_\kappa} q_\kappa [e^{C\cdot S_\kappa} \boldsymbol{\sigma}\cdot\boldsymbol{\psi}_\kappa - 2\sinh(C\cdot S_\kappa)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\psi}_\kappa]\,. \tag{5.25}$$

The Breit equation (4.10) for the matrix wave function is thus

$$\mathrm{LHS}_0 + \mathrm{LHS}_1 = \mathrm{RHS} \tag{5.26}$$

where the expressions are from Eqs. (5.11), (5.21), and (5.25).

Eq. (5.26) can be written explicitly as

$$\sum_{\kappa\lambda=0}^{3} q_\kappa \sigma_\lambda \Big\{ W_{\kappa\lambda}\psi_{\kappa\lambda} - (1-\delta_{\lambda 0})\bar{V}_\kappa \hat{r}_\lambda \sum_{\mu=1}^{3} \hat{r}_\mu \psi_{\kappa\mu} \Big\}$$

$$= \sum_{\kappa\lambda=0}^{3} \Big[\sum_{i=1}^{3} \Big\{ \alpha_i q_\kappa \sigma_\lambda + q_\kappa \sigma_\lambda \alpha_i \Big\} p_i + m_1 q_3 q_\kappa \sigma_\lambda - m_2 q_\kappa \sigma_\lambda q_3 \Big] \psi_{\kappa\lambda}\,, \tag{5.27}$$

where

$$W_{\kappa\lambda} = w\ e^{(A-3B-C)\cdot S_\kappa}\delta_{\lambda 0} + w\ (1-\delta_{\lambda 0}) e^{(A+B+C)\cdot S_\kappa}\,, \tag{5.28a}$$

$$\bar{V}_\kappa = 2w e^{(A+B)\cdot S_\kappa} \sinh(C\cdot S_\kappa)\,. \tag{5.28b}$$

Multiplying the equation from the right by $\sigma_\lambda q_\kappa$ and taking traces we finally get a set of 16 coupled equations for the wave function components,

$$\Big\{ W_{\kappa\lambda}\psi_{\kappa\lambda}\delta_{\lambda 0} - (1-\delta_{\lambda 0})\bar{V}_\kappa \hat{r}_\lambda \sum_{\mu=1}^{3} \hat{r}_\mu \psi_{\kappa\mu} \Big\}$$

$$= \sum_{\kappa'\lambda'=0}^{3} \sum_{i=1}^{3} (1 + f_{\kappa' 1} f_{\lambda' i}) g_{\kappa 1 \kappa'} g_{\lambda 1 \lambda'} p_i \psi_{\kappa'\lambda'} + \sum_{\kappa'=0}^{3} (m_1 - f_{\kappa' 3} m_2) g_{\kappa 3 \kappa'} \psi_{\kappa'\lambda}\,, \tag{5.29}$$

where

$$f_{\kappa i} = \delta_{\kappa 0} + (1-\delta_{\kappa 0})(2\delta_{\kappa i} - 1)\,, \tag{5.30}$$

and

$$g_{\kappa i \kappa'} = \delta_{\kappa 0}\delta_{i\kappa'} + \delta_{\kappa' 0}\delta_{i\kappa} + (1-\delta_{\kappa 0})(1-\delta_{\kappa' 0}) i\epsilon_{\kappa i \kappa'}\,. \tag{5.31}$$

The structure of these equations becomes more transparent by writing them out explicity in terms of the singlet and triplet wave functions

$$(\psi_{00}, \psi_{10}, \psi_{20}, \psi_{30}) = (\psi, \phi, i\chi, \eta), \tag{5.32}$$

$$(\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3) = (\boldsymbol{\psi}, \boldsymbol{\phi}, i\boldsymbol{\chi}, \boldsymbol{\eta}) \ : \tag{5.33}$$

$$[w - U_\psi]\psi = 2\boldsymbol{p} \cdot \boldsymbol{\phi} + (m_1 - m_2)\eta, \tag{5.34a}$$

$$[w - U_\phi]\phi = 2\boldsymbol{p} \cdot \boldsymbol{\psi} + (m_1 + m_2)\chi, \tag{5.34b}$$

$$[w - U_\chi]\chi = (m_1 + m_2)\phi, \tag{5.34c}$$

$$[w - U_\eta]\eta = (m_1 - m_2)\psi, \tag{5.34d}$$

$$[w - V_{\boldsymbol{\psi}} - \bar{V}_{\boldsymbol{\psi}}\hat{r}\hat{r}\cdot]\boldsymbol{\psi} = 2\boldsymbol{p}\phi + (m_1 - m_2)\boldsymbol{\eta}, \tag{5.34e}$$

$$[w - V_{\boldsymbol{\phi}} - \bar{V}_{\boldsymbol{\phi}}\hat{r}\hat{r}\cdot]\boldsymbol{\phi} = 2\boldsymbol{p}\psi + (m_1 + m_2)\boldsymbol{\chi}, \tag{5.34f}$$

$$[w - V_{\boldsymbol{\chi}} - \bar{V}_{\boldsymbol{\chi}}\hat{r}\hat{r}\cdot]\boldsymbol{\chi} = -2i\boldsymbol{p} \times \boldsymbol{\eta} + (m_1 + m_2)\boldsymbol{\phi}, \tag{5.34g}$$

$$[w - V_{\boldsymbol{\eta}} - \bar{V}_{\boldsymbol{\eta}}\hat{r}\hat{r}\cdot]\boldsymbol{\eta} = -2i\boldsymbol{p} \times \boldsymbol{\chi} + (m_1 - m_2)\boldsymbol{\psi}, \tag{5.34h}$$

where

$$w - U_\kappa = we^{(A-3B-C)\cdot S_\kappa}, \tag{5.35a}$$

$$w - V_\kappa = we^{(A+B+C)\cdot S_\kappa}, \tag{5.35b}$$

and $\bar{V}_\kappa$ is defined in Eq. (5.28).

In order to see explicitly the distinction between the traditional Breit approach and that of constraint dynamics, we perform an angular momentum decomposition. For the spin-zero wave functions we take

$$\psi = \psi_j Y_{jm},\ \phi = \phi_j Y_{jm},\ \chi = \chi_j Y_{jm},\ \eta = \eta_j Y_{jm}, \tag{5.36a}$$

where Y_{jm} is an ordinary spherical harmonic ($j = l$ here). For the spin-one wave functions, we take a form that depends on the spatial parity:

$$\boldsymbol{\phi} = \phi_{jy}\boldsymbol{Y_{jm}} + \phi_{jz}\boldsymbol{Z_{jm}},\ \text{or}\ = \phi_{jx}\boldsymbol{X_{jm}}; \tag{5.37a}$$

$$\boldsymbol{\psi} = \psi_{jy}\boldsymbol{Y_{jm}} + \psi_{jz}\boldsymbol{Z_{jm}},\ \text{or}\ = \psi_{jx}\boldsymbol{X_{jm}}; \tag{5.37b}$$

$$\boldsymbol{\eta} = \eta_{jy}\boldsymbol{Y_{jm}} + \eta_{jz}\boldsymbol{Z_{jm}},\ \text{or}\ = \eta_{jx}\boldsymbol{X_{jm}}; \tag{5.37c}$$

$$\boldsymbol{\chi} = \chi_{jy}\boldsymbol{Y_{jm}} + \chi_{jz}\boldsymbol{Z_{jm}},\ \text{or}\ = \chi_{jx}\boldsymbol{X_{jm}}; \tag{5.37d}$$

where the three orthonormal vector spherical harmonics are

$$\boldsymbol{X_{jm}} = \frac{\boldsymbol{L}}{\sqrt{j(j+1)}}Y_{jm},\ \boldsymbol{Y_{jm}} = \frac{\boldsymbol{r}}{r}Y_{jm},\ \boldsymbol{Z_{jm}} = i\frac{r\boldsymbol{p}}{\sqrt{j(j+1)}}Y_{jm}\,. \tag{5.38}$$

The first and last vanish for $j = 0$.

To obtain the radial wave equations, we use the following identities

$$\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{X_{jm}} = 0,\ \boldsymbol{p}\cdot\boldsymbol{X_{jm}} = 0, \tag{5.39a}$$

$$\frac{\boldsymbol{r}}{r}\times\boldsymbol{X_{jm}} = i\boldsymbol{Z_{jm}},\ \boldsymbol{p}\times\boldsymbol{X_{jm}} = \frac{\sqrt{j(j+1)}}{r}\boldsymbol{Y_{jm}} + \frac{\boldsymbol{Z_{jm}}}{r}, \tag{5.39b}$$

$$\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{Y_{jm}} = Y_{jm},\ \boldsymbol{p}\cdot\boldsymbol{Y_{jm}} = -\frac{2i}{r}Y_{jm}, \tag{5.39c}$$

$$\frac{\boldsymbol{r}}{r}\times\boldsymbol{Y_{jm}} = 0,\ \boldsymbol{p}\times\boldsymbol{Y_{jm}} = -\frac{\sqrt{j(j+1)}}{r}\boldsymbol{X_{jm}}, \tag{5.39d}$$

$$\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{Z_{jm}}=0,\ \boldsymbol{p}\cdot\boldsymbol{Z_{jm}}=i\frac{\sqrt{j(j+1)}}{r}Y_{jm}, \tag{5.39e}$$

$$\frac{\boldsymbol{r}}{r}\times\boldsymbol{Z_{jm}}=\boldsymbol{X_{jm}},\ \boldsymbol{p}\times\boldsymbol{Z_{jm}}=\frac{\boldsymbol{X_{jm}}}{r}. \tag{5.39f}$$

The equations so obtained can be divided into two sets of different total parity. One set has the natural parity solution:

$$\mathcal{P}\Psi_{jm}=(-)^{j+1}\Psi_{jm}, \tag{5.40}$$

and involves the 8 wave functions:

$$\phi=\phi_j Y_{jm},\ \chi=\chi_j Y_{jm}, \tag{5.41a}$$

$$\boldsymbol{\phi}=\phi_{jx}\boldsymbol{X_{jm}},\ \boldsymbol{\psi}=i\psi_{jy}\boldsymbol{Y_{jm}}+i\psi_{jz}\boldsymbol{Z_{jm}}, \tag{5.41b}$$

$$\boldsymbol{\chi}=\chi_{jx}\boldsymbol{X_{jm}},\ \boldsymbol{\eta}=i\eta_{jy}\boldsymbol{Y_{jm}}+i\eta_{jz}\boldsymbol{Z_{jm}}. \tag{5.41c}$$

(Four of the wave functions $\phi_{jx},\psi_{jz},\chi_{jx}$ and η_{jz} do not appear for $j=0$, because their angular parts vanish.) The eight simultaneous equations satisfied by them are

$$(w-U_\phi)\phi_j=2\psi'_{jy}+\frac{4}{r}\psi_{jy}-2\frac{\sqrt{j(j+1)}}{r}\psi_{jz}+(m_1+m_2)\chi_j, \tag{5.42a}$$

$$(w-V_{\boldsymbol{\psi}}-\bar{V}_{\boldsymbol{\psi}})\psi_{jy}=-2\phi'_j+(m_1-m_2)\eta_{jy}, \tag{5.42b}$$

$$(w-V_{\boldsymbol{\eta}})\eta_{jz}=-2\chi'_{jx}-\frac{2}{r}\chi_{jx}+(m_1-m_2)\psi_{jz}, \tag{5.42c}$$

$$(w-V_{\boldsymbol{\chi}})\chi_{jx}=2\eta'_{jz}+\frac{2}{r}\eta_{jz}-2\frac{\sqrt{j(j+1)}}{r}\eta_{jy}+(m_1+m_2)\phi_{jx}. \tag{5.42d}$$

$$(w-U_\chi)\chi_j=(m_1+m_2)\phi_j, \tag{5.43a}$$

$$(w-V_{\boldsymbol{\psi}})\psi_{jz}=-2\frac{\sqrt{j(j+1)}}{r}\phi_j+(m_1-m_2)\eta_{jz}, \tag{5.43b}$$

$$(w - V_{\boldsymbol{\phi}})\phi_{jx} = (m_1 + m_2)\chi_{jx}, \tag{5.43c}$$

$$(w - V_{\boldsymbol{\eta}} - \bar{V}_{\boldsymbol{\eta}})\eta_{jy} = -2\frac{\sqrt{j(j+1)}}{r}\chi_{jx} + (m_1 - m_2)\psi_{jy}\,. \tag{5.43d}$$

The second set has the "unnatural" parity

$$\mathcal{P}\Psi_{jm} = (-)^j\Psi_{jm}, \tag{5.44}$$

and involves the 8 wave functions:

$$\psi = \psi_j Y_{jm},\ \eta = \eta_j Y_{jm}, \tag{5.45a}$$

$$\boldsymbol{\psi} = \psi_{jx}\boldsymbol{X_{jm}},\ \boldsymbol{\phi} = i\phi_{jy}\boldsymbol{Y_{jm}} + i\phi_{jz}\boldsymbol{Z_{jm}}, \tag{5.45b}$$

$$\boldsymbol{\eta} = \eta_{jx}\boldsymbol{X_{jm}},\ \boldsymbol{\chi} = i\chi_{jy}\boldsymbol{Y_{jm}} + i\chi_{jz}\boldsymbol{Z_{jm}}. \tag{5.45c}$$

The eight simultaneous equations satisfied by them are

$$(w - U_\psi)\psi_j = 2\phi'_{jy} + \frac{4}{r}\phi_{jy} - 2\frac{\sqrt{j(j+1)}}{r}\phi_{jz} + (m_1 - m_2)\eta_j, \tag{5.46a}$$

$$(w - V_{\boldsymbol{\chi}})\chi_{jz} = -2\eta'_{jx} - \frac{2}{r}\eta_{jx} + (m_1 + m_2)\phi_{jz}, \tag{5.46b}$$

$$(w - V_{\boldsymbol{\eta}})\eta_{jx} = 2\chi'_{jz} + \frac{2}{r}\chi_{jz} - 2\frac{\sqrt{j(j+1)}}{r}\chi_{jy} + (m_1 - m_2)\psi_{jx}. \tag{5.46c}$$

$$(w - V_{\boldsymbol{\phi}} - \bar{V}_{\boldsymbol{\phi}})\phi_{jy} = -2\psi'_j + (m_1 + m_2)\chi_{jy}, \tag{5.46d}$$

$$(w - U_\eta)\eta_j = (m_1 - m_2)\psi_j, \tag{5.47a}$$

$$(w - V_{\boldsymbol{\psi}})\psi_{jx} = (m_1 - m_2)\eta_{jx}, \tag{5.47b}$$

$$(w - V_{\boldsymbol{\phi}})\phi_{jz} = -2\frac{\sqrt{j(j+1)}}{r}\psi_j + (m_1 + m_2)\chi_{jz}, \tag{5.47c}$$

$$(w - V_{\boldsymbol{\chi}} - \bar{V}_{\boldsymbol{\chi}})\chi_{jy} = -2\frac{\sqrt{j(j+1)}}{r}\eta_{jx} + (m_1 + m_2)\phi_{jy}, \tag{5.47d}$$

where the terms $w - U$, $w - V$ are given in Eqs. (5.35a-b) and $w - V - \bar{V}$ by

$$w - V_\kappa - \bar{V}_\kappa = we^{(A+B-C)\cdot S_\kappa}. \tag{5.48}$$

Each set of 8 equations consists of 4 differential equations [Eqs. (5.42) or (5.46)] and 4 algebraic equations [Eqs. (5.43) or (5.47)]. The algebraic equations can be used to express 4 of the 8 wave functions in terms of the remaining 4. For example, two of the eliminated wave functions are

$$\chi_j = \frac{m_1 + m_2}{w - U_\chi}\phi_j\ , \tag{5.49}$$

$$\eta_j = \frac{m_1 - m_2}{w - U_\eta}\psi_j\ \ . \tag{5.50}$$

After the elimination, pole singularities could appear in the differential equations at particle separations r where equations such as Eq. (5.49) have poles, e.g., where

$$w - U_\chi = 0;\ w - U_\eta = 0,\ m_1 \neq m_2 \tag{5.51}$$

provided that the total relativistic center-of-mass energy w is nonzero. These are the well-known singularities that plague the traditional Breit equations.

However, from the perspective of the exponentiated interactions of constraint dynamics such as that shown in Eq. (5.35), these structural poles can appear only if the potential generators in the exponent go to $-\infty$. In the absence of such singular behavior, the constraint two-body equations, or their Breit analogs, are free of the pathologies described in [6,7]. These pathologies arise because the wave functions appearing on the right-hand side of Eq. (5.49) must have zeros at the pole positions. These additional boundary conditions give rise to spurious resonances in the continuum for sufficiently strong interactions, and to spurious bound states for any nonzero interaction strength at total energies (including rest masses) which go to zero.

We should add, for the sake of completeness, that another class of spurious solutions appears when the values of the exponential generators go to $+\infty$. Then the algebraic equations impose the new boundary conditions that the wave functions appearing on the left-hand sides of these equations must vanish at these singular points.

It is important to point out that if the exponentiated potentials given in Eqs. (5.35) and (5.48) are approximated by finite Taylor approximants, it might be possible to satisfy the singularity conditions such as (5.50) whenever the approximant has a zero where the true value of the exponential function is nonzero. In other words, it is the exponential structure of the effective interaction that protects the CTB Dirac equations from the undesirable pathology. We trace that exponential structure to the hyperbolic forms (3.3) of the constraint equations which in turn are motivated by the requirements of compatibility. Recall that in those forms the "generators" Δ produce potential terms in the "external potential" form of the constraint equations which have the expected transformation properties, including the essential recoil corrections.

Our result can also be interpreted in another way. The exponential functions appearing on the left hand side of Eq. (4.10) can be expanded in terms of hyperbolic functions. Indeed, one could express the "effective potentials" contained in $exp(2\mathcal{D})$ in the form given by the right-hand side of Eq. (4.11). If we now parametrize these effective potentials directly using functions with no singularity at finite r, we will find that under favorable circumstances the singularity condition (5.50) can still be satisfied. This is in fact the result of [6,7]. Thus the regularity of the effective potentials appearing in $exp(2\mathcal{D})$ does not guarantee the regularity of the resulting Breit equation. It is the regularity of the basic potentials appearing in the exponent $2\mathcal{D}$ that guarantees the regularity of the resulting Breit equation.

VI. COMPARISON OF THE CONSTRAINT AND BREIT EQUATIONS FOR QED

In this section we discuss the implications of our new Breit-like equation in quantum electrodynamics. Consider first the original Breit equation whose matrix form is

$$(w+\frac{\alpha}{r})\Psi' + a\frac{\alpha}{r}\boldsymbol{\alpha}\cdot\Psi'\boldsymbol{\alpha} + b\frac{\alpha}{r}\boldsymbol{\alpha}\cdot\hat{r}\Psi'\boldsymbol{\alpha}\cdot\hat{r} = \boldsymbol{p}\cdot\boldsymbol{\alpha}\Psi' + \boldsymbol{p}\cdot\Psi'\boldsymbol{\alpha} + m_1\beta\Psi' - m_2\Psi'\beta\,, \qquad (6.1)$$

where a and b are potential parameters. By identifying it with the center-of-mass form of the Breit-like form Eq. (4.10) of our covariant constraint equation, we can solve for the generators given by Eq. (5.5). Using Eqs. (5.26) or equivalently Eqs. (5.27-28) we obtain the following twelve equations for the potentials.

$$exp[(A-3B-C)\cdot S_0] = 1+(1+3a+b)\zeta, \qquad (6.2a)$$

$$exp[(A-3B-C)\cdot S_1] = 1+(1+3a+b)\zeta, \qquad (6.2b)$$

$$exp[(A-3B-C)\cdot S_2] = 1+(1-3a-b)\zeta, \qquad (6.2c)$$

$$exp[(A-3B-C)\cdot S_3] = 1+(1-3a-b)\zeta, \qquad (6.2d)$$

$$exp[(A+B+C)\cdot S_0] = 1+(1-a-b)\zeta, \qquad (6.2e)$$

$$exp[(A+B+C)\cdot S_1] = 1+(1-a-b)\zeta, \qquad (6.2f)$$

$$exp[(A+B+C)\cdot S_2] = 1+(1+a+b)\zeta, \qquad (6.2g)$$

$$exp[(A+B+C)\cdot S_3] = 1+(1+a+b)\zeta, \qquad (6.2h)$$

$$exp[(A+B-C)\cdot S_0] = 1+(1-a+b)\zeta, \qquad (6.2i)$$

$$exp[(A+B-C)\cdot S_1] = 1+(1-a+b)\zeta, \tag{6.2j}$$

$$exp[(A+B-C)\cdot S_2] = 1+(1+a-b)\zeta, \tag{6.2k}$$

$$exp[(A+B-C)\cdot S_3] = 1+(1+a-b)\zeta, \tag{6.2l}$$

in which $\zeta = \alpha/(wr)$. The first four equations come from the singlet wave function ($\lambda = 0$ terms in Eq.(5,28a)), while the last eight arise from the triplet, with the last four coming from a combination of Eq. (5.28b) and the triplet (or $\lambda \neq 0$) part of Eq. (5.28a). These twelve algebraic equations can be solved for the twelve unknown generators shown in Eq. (5.5). One can readily show that six of these generators vanish (corresponding to scalar, pseudoscalar, and tensor interactions):

$$S = T = \mathcal{F} = Y = C = L = 0. \tag{6.3}$$

The remaining generators, J, H, I, G, N and R, are nonzero, but we shall not need them in our discussion except in the combinations appearing on the left-hand sides of the algebraic equations, Eqs. (5.43) and (5.47). These are just the six equations shown in Eqs. (6.2c-f) and (6.2k,l), now expressible as

$$w - U_\chi = w - U_\eta = w\ exp[J + 3I - N - H + 3\mathcal{G} + R] = 1 + (1-3a-b)\zeta\,, \tag{6.4}$$

$$w - V_{\boldsymbol{\psi}} = w - V_{\boldsymbol{\phi}} = w\ exp[J - I + N + H + \mathcal{G} + R] = 1 + (1-a-b)\zeta\,, \tag{6.5}$$

$$w - V_{\boldsymbol{\chi}} - \bar{V}_{\boldsymbol{\chi}} = w - V_{\boldsymbol{\eta}} - \bar{V}_{\boldsymbol{\eta}} = w\ exp[J - I + N - H + \mathcal{G} - R] = 1 + (1+a-b)\zeta\,. \tag{6.6}$$

Singularities at finite particle separation r arise, for nonzero w, when the right-hand sides of these equations vanish (assuming that the numerators in the algebraic equations in which these occur do not). Recalling the context in which these equations appear, we see that this occurs when

$$3a + b > 1 \tag{6.7}$$

for all $\mathcal{P} = (-)^{j+1}$ states, and all $\mathcal{P} = (-)^j$ states when $m_1 \neq m_2$;

$$a + b > 1 \tag{6.8}$$

for all $\mathcal{P} = (-)^j$ states, all $\mathcal{P} = (-)^{j+1}$ states when $m_1 \neq m_2$; and for $j = 0$ states when $m_1 = m_2$ and

$$b - a > 1 \tag{6.9}$$

for all $\mathcal{P} = (-)^{j+1}$ states when $m_1 \neq m_2$, and for $j = 0$ states when $m_1 = m_2$ and all $\mathcal{P} = (-)^j$ states. This confirms the result first derived in Ref. [7].

It is worth noting that the original Breit equation corresponds to $a = b = 1/2$ while the Barut equation corresponds to $a = 1, b = 0$ [21]. Both equations are therefore singular according to the first of the above three conditions.

From the perspective of the exponential generators of constraint dynamics, the vanishing of the right-hand sides of Eqs. (6.4-6.6) is possible only when the generators take on the value $-\infty$ at finite r. Hence they can be avoided by simply using generators not having such r singularities. Indeed, in their constraint approach to QED, Crater and Van Alstine [11,12] found that

$$e^{\mathcal{G}} = e^{-J} = (1 + 2\alpha/(wr))^{-1/2} \tag{6.10}$$

with all of the remaining ten generators set equal to zero. As a result, no midrange zero appears since

$$w - U_\chi = w \ \exp(J + 3\mathcal{G}) = w \ \exp(2\mathcal{G}) = \frac{1}{1 + 2\alpha/wr} \tag{6.11}$$

is finite for positive r. Furthermore, the Lorentz nature of the two nonzero generators matches that of the vector character of the QED interaction at this level. In contrast, the Breit generators involve pseudovector as well as vector parts and additional vector and pseudovector "tensor terms" corresponding to the Coulomb gauge (as opposed to the Feynman gauge used in the constraint equations).

The conclusions obtained in this section for QED can be extended to other interactions (scalar, pseudoscalar, psuedovector and tensor) important in semiphenomenological applications in nuclear and particle physics. Thus, one may with safety solve the new Breit equations numerically with no concern about structural singularities which would otherwise render such solutions meaningless.

VII. CONCLUSIONS AND DISCUSSIONS

Breit equations and constraint Dirac equations of relativistic two-body quantum mechanics have markedly different properties: Breit equations could have structural singularities at finite particle separations even when the interactions themselves are nonsingular there, while constraint Dirac equations seem to be free of them. CTB Dirac equations are manifestly covariant, while Breit equations are not, being valid only in the center-of-mass frame. We are able to understand, and to reconcile, their differences in this paper.

The constraint Dirac equations were originally derived for scalar interactions with the help of supersymmetries in addition to Dirac's constraint dynamics. Generalizing this concept to arbitrary interactions leads to a "hyperbolic" form of the equations. The two single-particle Dirac equations can then be recast into two other equations: (1) a covariant Breit-like equation with exponentiated interactions, and (2) a covariant equation describing an additional orthogonality condition on the relative momentum. We use the equivalence between these two types of equations to show that the constraint Dirac equations are completely free of the unphysical structural singularities when the exponential structure of the interactions are not tampered with.

The advantage of the constraint form of the Breit equation is that the structural singularities of the traditional Breit equation are now moved entirely to the exponential generators of the interaction. As a consequence, they can be eliminated right from the beginning by the simple requirement that these generators themselves be free of singularities at finite separations. The resulting Breit equations are then guaranteed to be free of the undesir-

able structural singularities that plague traditional Breit equations. These improved Breit equations, which are dynamically equivalent to the constraint Dirac equations, can now be used in nonperturbative descriptions of highly relativistic and strong-field problems such as those appearing in two-body problems in quantum electro- and chromo-dynamics, an in nucleon-nucleon scattering.

Of course, the constraint Dirac equations can also be used, now that we know how to keep them singularity-free. However, in actual applications, they have to be reduced down to a set of coupled differential equations before actual solutions can be attempted. These differential equations are transforms (see Eq.(4.9)) of those obtained from the equivalent Breit equations. So in reality the two different formulations have now become completely identical to each other.

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